I. Introduction to Limits

1. An important question: Given a function f, and a number a. As x gets closer and closer to a, but x does not equal a, does f(x) get closer and closer to some number L?

   If it does, then we say "the limit of f(x), as x approaches a, is equal to L", and we write $\lim_{x\to a} f(x) = L$.

   It is important to solve limit problems **numerically, graphically,** and **analytically**.

2. Example: Given $f(x) = \frac{x-1}{x^2+x-2}$. As $x \to 1$, does $f(x) \to$ _______?

On your calculator, set a "decimal window" (ZOOM, 4:ZDecimal) and let $Y_1 = \frac{x-1}{x^2+x-2}$.

a. Graph f and discuss the result.

b. Evaluate $Y_1(0.9)$, $Y_1(0.99)$, $Y_1(0.999)$, and $Y_1(1.1)$, $Y_1(1.01)$, $Y_1(1.001)$.

c. Graphically (and numerically) it appears that the $\lim_{x\to 1} \frac{x-1}{x^2+x-2} =$ ________.

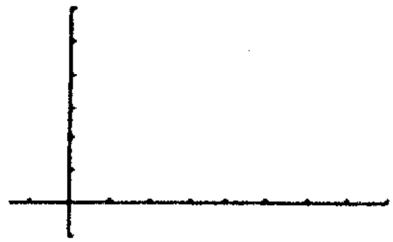
d. Evaluate $\lim_{x\to 1} \frac{x-1}{x^2+x-2}$ analytically.

$$\lim_{x\to 1} \frac{x-1}{x^2+x-2} =$$

e. What about $\lim_{x\to -2} \frac{x-1}{x^2+x-2}$? ____________________

3. Problem: Given $f(x)=(1+x)^{2/x}$

   a. Sketch a graph of the function for $x > -1$. Show all asymptotes with dotted lines and other undefined values with an "open circle".



   b. Estimate the $\lim_{x\to 0} f(x)$ by evaluating f for values close to 0. Approximate the limit to 4 decimal places.

   $\lim_{x\to 0}(1+x)^{2/x} \approx$ ____________________

   c. Do you know the **exact value** of $\lim_{x\to 0}(1+x)^{2/x}$? __________

II. Continuity

1. To develop the definition of continuity at a point, have students:

   "Draw an example of a function that is not continuous (has a "break") at a number $x = c$. Draw as many different kinds of discontinuities at the number $x = c$ as you can."

Discuss what makes these functions discontinuous at the number $x = c$.

2. Worksheet - Limits and Continuity

3. My Favorite Function!

   Consider the function $f(x) = x + [\![\cos(\pi x)]\!]$

   Note: $f(x)=[\![x]\!]$ is called the "Greatest Integer Function" (or the "Floor Function") and can be found on most calculators as the **int(x)**.

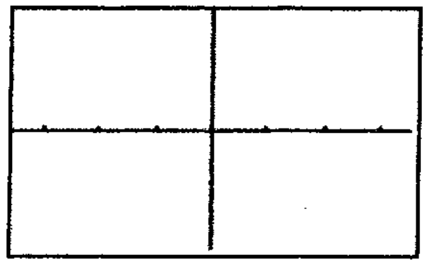
   a. Sketch the graph of f using a Decimal Window, discuss the values of x that make f discontinuous, and explain, using the definition of continuous functions, why f is not continuous at these values.

III. The Derivative

1. Introduction to the Derivative - The Tangent Line Problem

Given $f(x) = x^2 - 3x$.

a. Sketch the graph of f. Verify with your calculator.



b. Find the <u>slope of the secant line</u> to $f(x) = x^2 - 3x$ passing through the points when $x = 1$ and $x = 3$.

c. Find the <u>slope of the tangent line</u> to $f(x) = x^2 - 3x$ passing through the point when $x = 1$.
(Note: We need to approximate the slope. How can we do that?)

On calculator, enter: $Y_1 = x^2 - 3x$

On Home Screen, enter $\dfrac{Y_1(1.01) - Y_1(1)}{1.01 - 1}$

Conjecture: As the "second point" gets closer to $x = 1$, the slope of the secant line approaches the slope of the tangent line. Therefore, we can write:

$$m_{tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h} = f'(1)$$

d. Find the slope of the tangent line to $f(x) = x^2 - 3x$ at $x = 4$.

At $x = 4$, m_{tan} = _________

e. Built into your calculator is a feature that will estimate the derivative of a function at a value. From the Home screen of your calculator, press the MATH key, and select 8:nDeriv(. (This stands for a "numerical derivative".)

The parameters for "nDeriv" are:

nDeriv(function , variable , value)

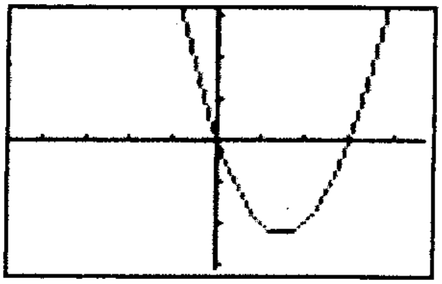
For our example, we should write: ____________________________________

f. Use the "NDeriv" to find f'(a) for the values of a in the chart below if $f(x) = x^2 - 3x$

a	-1	0	1	2	3	4
f'(a)						

g. Do you see a pattern in the chart above? In other words, for any x, f'(x) = _______________

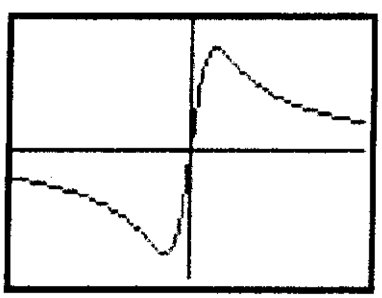
h. Sketch the graph of f' with the graph of f. Does the graph of f give you any information about the graph of f'?

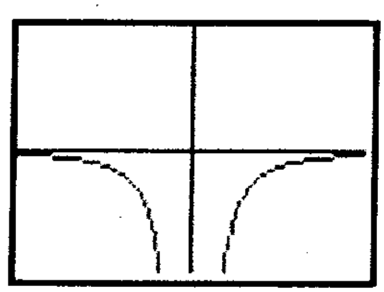


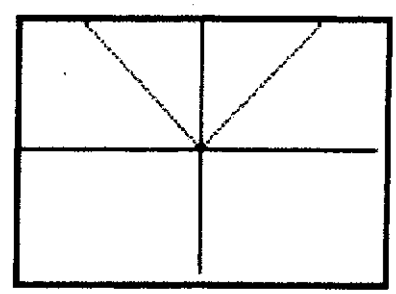
2. Graphs of f and f'.

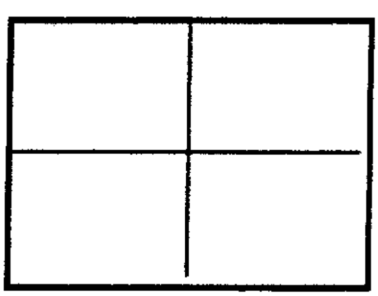
Note: I like to introduce this concept very early in the development of the derivative concept. We are graphing derivative functions before we know any rules for finding derivatives analytically.

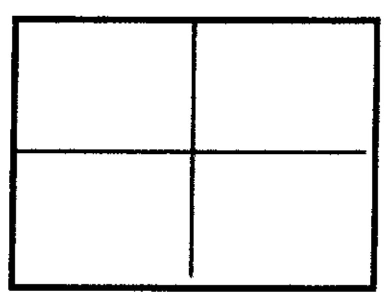
Given the graph of the function shown, sketch the graph of the derivative function f' directly below it. Remember, **"the y value on the graph of f' is the slope of the tangent line to the graph of f."**

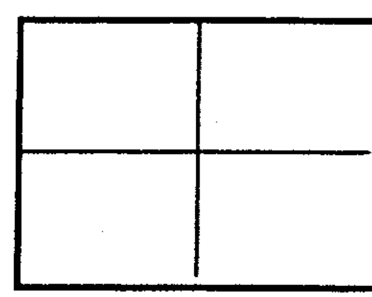












3. Worksheet - Graphs of f and f'

IV. Derivative Properties

1. Worksheet - Discovering Derivative Properties

2. Derivatives of Exponential Functions

Notes:

a. Traditionally, the $\frac{d}{dx}(e^x)$ is evaluated first, then generalized to the $\frac{d}{dx}(a^x)$.

I prefer to discover the $\frac{d}{dx}(a^x)$ first, then look at $\frac{d}{dx}(e^x)$ as a special case.

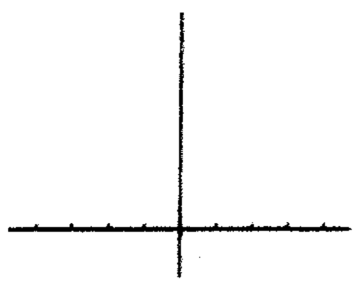
b. This is very difficult to prove analytically using the definition of the derivative. You get a limit expression such as:

$$f'(x) = a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h}$$

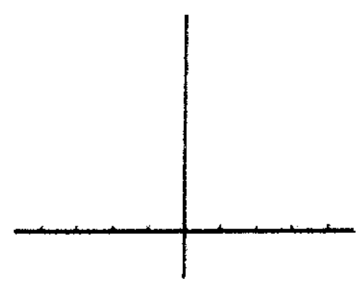
c. We should know that ln1 = 0, lne = 1, but need a calculator to evaluate ln2. ln2 = ____________

d. If $f(x) = 2^x$, sketch the graphs of f(x) and f'(x).

f



f'



e. Let: $y_1 = 2^x$ .and $y_2 = \frac{d}{dx}(2^x)$. Graph the functions. (Note: Y2 = nDeriv(Y1 , X , X))

f. What is the relationship between y1 and y2? It looks like $y_2 = k \cdot y_1$, where $0 < k < 1$. To verify this, and to find k, let $y_3 = \frac{y_2}{y_1}$, and graph it.

k = ____________________

Therfore, $y_2 = \frac{d}{dx}(2^x) =$ ________________ = ________________

g. **Conclusions:**

1. $\frac{d}{dx}(2^x) =$ ____________ 2. In general, $\frac{d}{dx}(a^x) =$ ______________

3. And, if a = e, then $\frac{d}{dx}(e^x) =$ ______________

3. Derivatives of Sine and Cosine functions (graphically).

4. Discovering the Chain Rule

Make a guess.

a. We know: If $y=2\sin x$, then $y'=$ ______________________________

Guess: If $y=2\sin(5x+1)$, then $y'=$ ______________________________

b. We know: If $y = x^3$, then $y'=$ ______________________________

Guess: If $y = (3x^2+1)^3$, then $y'=$ ______________________________

c. We know: If $y = \sqrt{x}$, then $y'=$ ______________________________

Guess: If $y = \sqrt{\cos x}$, then $y'=$ ______________________________

Checking the guesses.

a. Check the guess for Part a above <u>graphically</u>.

Let: Y1= 2sin(5x+1)

Y2= ______________________ (The guess)

Y3= nDeriv(Y1,X,X)

Deactivate Y1 and graph Y2 and Y3.

Was the guess correct? __________

The correct answer is: If $y=2\sin(5x+1)$, then $y'=$ ______________________________

b. Check the guess for Part b above algebraically.

If $y = (3x^2+1)^3$, expand the right side of the equation, then find the derivative.

c. If we see the pattern, correct the guess for Part c, and check the answer graphically.

If $y = \sqrt{\cos x}$, then y'=________________________

Let: $Y1 = \sqrt{\cos x}$

Y2= ____________________ (New guess for y')

Y3= nDeriv(Y1,X,X)

Deactivate Y1 and graph Y2 and Y3.

This property for finding derivatives of a composition of functions is called the **Chain Rule.**

It says:

If y = f(g(x)), then y'= ____________________________

5. The Product Rule (Worksheet) and Quotient Rule

6. Worksheet - Discovering Derivatives of the Other Trigonometric Functions

7. Curve Sketching

It's always nice to have different and interesting functions to use for your "curve sketching" problems. Here are a few that I have found.

a. $f(x) = \sqrt[3]{x^2 - 2x}$

b. $f(x) = 2x^2 e^x$

c. $f(x) = \sin^2 x$

d. $f(x) = \ln(\cos x)$ (Be careful of the domain!)

V. Applications of the Derivative

1. Calculus Application Problem - Heat It, Then Cool It
2. Calculus Application Problem - TICTOC
3. Calculus Application Problem - Keep On Folding
4. Calculus Application Problem - Around the Corner

VI. Implicit Differentiation

1. Introduce with the following problems:

a. $\frac{d}{dx}(3x + 4y) =$ ____________________

b. $\frac{d}{dx}(3x^3 + 4y^2) =$ ____________________

c. $\frac{d}{dx}(\cos x + \sin y) =$ ____________________

2. A nice example to use when developing implicit differentiation.

Given $x^2 - 4y^2 = 16$

a. Do you know what this graph looks like? ______________________

b. Solve for y and graph the functions on your calculator.

c. Find y', and then find the slope of the tangent line at $\left(4\sqrt{2}, -2\right)$.

d. Use implicit differentiation to find y', and then find the slope of the tangent line at $\left(4\sqrt{2}, -2\right)$.

3. An example: Given the relation defined as $x^2 + xy + y^2 = 7$.

a. Use **implicit differentiation** to find the derivative y'.
(Hint: Don't forget to use the Product Rule on the "xy" term.)

b. Find the **slope of the tangent line** to the curve at the point (-2 , -1). ______________

c. Solve the original equation for y in terms of x and graph the function(s). Show your graph below.

(Hint: To solve for y, you need to use the quadratic formula.

Let a = _____ , b = _____ , c= _______)

d. Use your calculator (and your graph) to evaluate the derivative of the function that you graphed in part c above at the point (-2 , -1) to check your answer in part b.

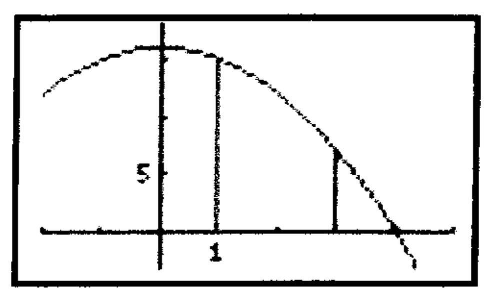
e. Write the **equation of the tangent line** to $x^2 + xy + y^2 = 7$ at (-2 , -1). Graph the tangent line on your calculator and show the line on the graph above.

4. Other interesting implicitly defined relations.

a. $y^2(2-x)=x^3$ at (1 , -1) b. $x^2y+xy^2=6$ at (1 , -3) c. $2e^{xy}-x=0$ at (2 , 0)

VII. Introduction to the Definite Integral - The Area Problem

1. Example: Shown to the right is the region bounded by the function $f(x)=16-x^2$, the x-axis, and the lines x = 1 and x = 3.



Write and evaluate an expression we could use to approximate the area under $f(x)=16-x^2$ from x = 1 to x = 3 using:

a. 4 rectangles

b. 10 rectangles

c. 50 rectangles

d. n rectangles. (We can not evaluate this expression without a CAS, but it is important that we are able to write it.

e. How can we find the exact area of the region?

2. Another example: Given the region bounded by $f(x)=2^x$, the x-axis, and the vertical lines $x = -2$ and $x = -4$.

a. Sketch the region described.

b. Write and evaluate (with your calculator) an expression that we could use to aproximate the area under $f(x)=2^x$ from $x = -2$ to $x = 4$ using:

i. 6 rectangles, with their heights determined by the right endpoint of each subinterval (without using sigma notation). Show these 6 rectangles on your graph.

ii. 20 retangles with their heights determined by the right endpoint of each subinterval (using sigma notation).

iii. n rectangles (We can not evaluate this expression.)

3. Definition of the the Definite Integral - The Area Problem Generalized

Given a function f, such that $f(x) \geq 0$, and the lines $x = a$ and $x = b$. Write an expression in sigma notation that could be used to find the area of the region bounded by f, the x-axis, $x = a$ and $x = b$.

d. Use your calculator (and your graph) to evaluate the derivative of the function that you graphed in part c above at the point (-2 , -1) to check your answer in part b.

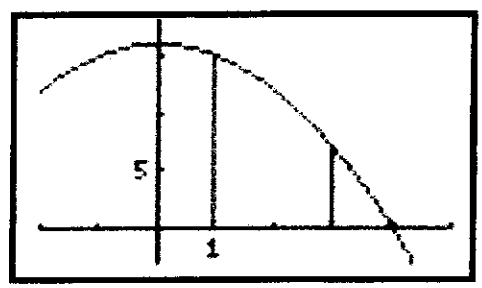
e. Write the **equation of the tangent line** to $x^2 + xy + y^2 = 7$ at (-2 , -1). Graph the tangent line on your calculator and show the line on the graph above.

4. Other interesting implicitly defined relations.

a. $y^2(2-x) = x^3$ at (1 , -1)   b. $x^2y + xy^2 = 6$ at (1 , -3)   c. $2e^{xy} - x = 0$ at (2 , 0)

VII. Introduction to the Definite Integral - The Area Problem

1. Example: Shown to the right is the region bounded by the function $f(x) = 16 - x^2$, the x-axis, and the lines x = 1 and x = 3.



Write and evaluate an expression we could use to approximate the area under $f(x) = 16 - x^2$ from x = 1 to x = 3 using:

a. 4 rectangles

b. 10 rectangles

c. 50 rectangles

d. n rectangles. (We can not evaluate this expression without a CAS, but it is important that we are able to write it.

e. How can we find the exact area of the region?

2. Another example: Given the region bounded by $f(x) = 2^x$, the x-axis, and the vertical lines x = -2 and x = -4.

a. Sketch the region described.

b. Write and evaluate (with your calculator) an expression that we could use to aproximate the area under $f(x) = 2^x$ from x = -2 to x = 4 using:

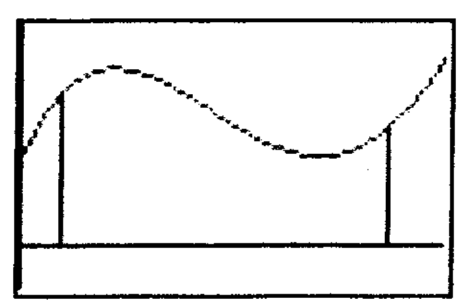
i. 6 rectangles, with their heights determined by the right endpoint of each subinterval (without using sigma notation). Show these 6 rectangles on your graph.

ii. 20 retangles with their heights determined by the right endpoint of each subinterval (using sigma notation).

iii. n rectangles (We can not evaluate this expression.)

3. Definition of the the Definite Integral - The Area Problem Generalized

Given a function f, such that $f(x) \geq 0$, and the lines x = a and x = b. Write an expression in sigma notation that could be used to find the area of the region bounded by f, the x-axis, x = a and x = b.



4. Evaluating Definite Integrals on the Calculator

This can be done one of two ways with our calculator.

a From the Home Screen, press the MATH key and select "9:fnInt". (This stands for a "function numerical integral".)

Then to evaluate $\int_{-2}^{4} 2^x \, dx$, enter:

fnInt(______________________________) = ______________

b. If the function has been graphed on your calculator with the interval included, from the graph select CALC, and 7: $\int f(x)dx$, and enter the limits of integration.

5 Program **DrawRec** on calculator

6. Worksheet - Evaluating Definite Integrals with Geometry

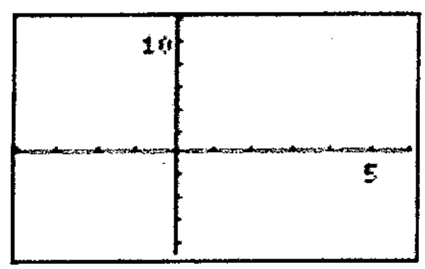
7. Worksheet - Properties of the Definite Integral

VIII. The Fundamental Theorem of Calculus

1. Introduction to the Fundamental Theorem of Calculus

Let $f(t) = -2t + 4$

Graph f on [-4 , 6].



Use geometry to evaluate the following integrals. Check your answer with your calculator.

a. $\int_{-2}^{-2} (-2t + 4)dt =$ ______________ b. $\int_{-2}^{0} (-2t + 4)dt =$ ______________

c. $\int_{-2}^{2} (-2t + 4)dt =$ ______________ d. $\int_{-2}^{4} (-2t + 4)dt =$ ______________

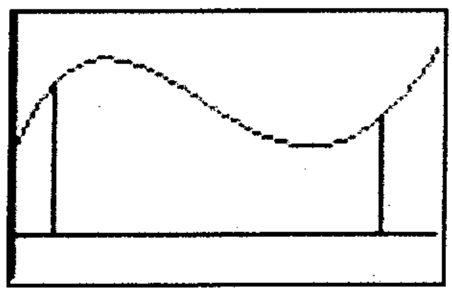
e. $\int_{-2}^{6} (-2t + 4)dt =$ ______________ f. $\int_{-2}^{-4} (-2t + 4)dt =$ ______________

Question: As the upper limit of integration in the definite integral changes, the value of the integral changes. Does this integral change values in some pattern that we can find?

Let $A(x)=\int_{-2}^{x}(-2t+4)dt$. Can we find a function rule for A(x)? We know that:

$A(-2)=0$, $A(0)=12$, $A(2)=16$, $A(4)=12$, $A(6)=0$, and $A(-4)=-20$

In general, given a function y=f(t). If $A(x)=\int_{a}^{x}f(t)dt$, can we find a function rule for A(x)?



2. Discovering the Fundamental Theorem of Calculus

a. Worksheet - Discovering the Fundamental Theorem of Calculus

b. Discussion: Where does the "constant" come from?

We know: $\int_{\frac{\pi}{2}}^{x}\cos t\,dt =$ ______________________

Graph the integral $\int_{\frac{\pi}{2}}^{x}\cos t\,dt =$ using fnInt on your calculator.

Can you find where the constant C comes from? ______________________

Therefore, $\int_{\frac{\pi}{2}}^{x}\cos t\,dt = =$ ______________________

Conclusion: In general, let F be an antiderivative of f. If $A(x)=\int_{a}^{x}f(t)dt$, then

$A(x)=$ ______________________

In the example at the very beginning of this discussion, we were trying to find a function $A(x)$, such that $A(x)=\int_{-2}^{x}(-2t+4)dt$ and $A(-2)=0$, $A(0)=12$, $A(2)=16$, $A(4)=12$, and $A(6)=0$.

According to our conclusion, $A(x)=\int_{-2}^{x}(-2t+4)dt$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

3.. The Fundamental Theorem of Calculus

If $g(x)=\int_{a}^{x}f(t)dt$, then $g'(x)=\frac{d}{dx}(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)$

$=\frac{d}{dx}(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

<u>Part One of FTC</u>: ____________________________________

Also, if $\int_{a}^{x}f(t)dt=F(x)-F(a)$, where F is an antiderivative of f, and, if we substitute <u>b</u> for <u>x</u> we can write:

$\int_{a}^{b}f(t)dt= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

And if we substitute <u>x</u> for <u>t</u>, we can write:

$\int_{a}^{b}f(x)dx= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

<u>Part Two of FTC</u>: $\int_{a}^{b}f(x)dx= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

XII. Differential Equations and Slope Fields

1. Worksheet - Introduction to Slopefields

2. An example of a differential equation problem.

Tin Can Leakage Problem: Suppose you fill a tall (topless) tin can with water, then punch a small hole near the bottom with an ice pick. The water leaks quickly at first, then more slowly as the depth of the water decreases. It can be shown that ***the rate at which the water leaks out is directly proportional to the square root of the depth.***

a. Write a differential equation stating the relationship between the rate of change of the depth and the depth of the water.

b. Suppose that initially the depth of the can is 12 cm and it is initially draining at 3 cm/min. Find the constant of proportionality.

c. Solve the differential equation to find the depth as a function of the time.

d. Use the information (in part b) to find the particular solution.

e. Plot the graph of the depth of the water as a function of time. Over what values of the time does the graph give reasonable values?

f. When will the can be half-empty?

XIII. Integration by Parts

Example: Given $f(x) = x \cdot \cos x$

1. Sketch a graph of f on [0 , π/2].

2. If we want to find the area under the curve, and above the x-axis, from $x = 0$ to $x = \pi/2$ by the FTC, that is evaluate $\int_0^{\frac{\pi}{2}} x \cdot \cos x dx$, at this time we can not do it, because we don't know the antiderivative of $x \cdot \cos x$. But, let's evaluate it with our calculator.

$\int_0^{\frac{\pi}{2}} x \cdot \cos x dx =$ ____________ (Any guesses about the exact value? ____________________)

But let's "play around" with some calculus and algebra, and see if we can find $\int x \cdot \cos x dx$.

Let's start with:

$y = x \cdot \sin x$

Now, $\int_0^{\frac{\pi}{2}} x \cdot \cos x dx =$

And, if you want to derive the **Integration by Parts Formula**,

Let $u = f(x)$ and $v = g(x)$.

Note: $du = f'(x)dx = \frac{du}{dx} \cdot dx$ and $dv = g'(x)dx = \frac{dv}{dx} \cdot dx$.

Then start with $y = u \cdot v$, and follow the same steps as the example!

The **Integration by Parts Formula**:

$\int u\,dv =$ ________________________________

One advantage of this formula, is that it now allows us to find antiderivatives of some basic functions for which we know the derivative, for example: $f(x) = \ln x$, $f(x) = \sin^{-1} x$, and $f(x) = \tan^{-1} x$

XIV. Logistic Growth, Finding Antiderivatives by Partial Fraction Decomposition, and Modeling the Spread of an Infectious Disease.

1. Introduction to Logistic Growth - Starbucks Problem. (Worksheet)

2. Discussion of finding antiderivatives using Partial Fraction Decomposition.

Example: Find $\int \frac{4}{4x^2 - 1} dx$

3. Calculus Application Problem - Don't Catch It! (Worksheet)

XV. Infinite Sequences and Series

1. Comment: After discussing sequences (convergence/divergence), and an introduction to series (with partial sums and geometric series), move right into power series. Don't bother with series of constants and convergence/divergence tests. Concentrate on getting polynomials (then series as an extension) from a function

Examples: $\frac{1}{1-x}$, e^x, cosx, sinx, and lnx

Then look at how to get other series by "operating" on these series by differentiating, integrating, etc.

Examples: $x \cdot \sin x$, e^{-x^2}, $\frac{1}{1+x^2}$, and $\tan^{-1} x$ (by integrating $\frac{1}{1+x^2}$)

Use these series to motivate the need for the traditional convergence/divergence tests to determine if these series diverge/converge at the endpoints of the interval of convergence. (Although I never could figure out why it is important to determinre if the series converge at the endpoints. It is the interval that's important!)

2. It's fun to ask your students "What are the 5 most important (or most used) constants in mathematics?"

Answer: ______________________

What is the equation that you can write that uses all five of these constants?

Then use the power series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \cdot x^{2n}, \text{ and } \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \cdot x^{2n+1}$$

to prove the above equation.

3. Data Collection Activity - Let It Hang! (Worksheet)

XVI. Calculus and Parametric Equations

1. Introduction: Using parametric equations, $x = f(t)$ and $y = g(t)$. Plot (x, y) or $(f(t), g(t))$.

   Example: Graph $x = t + 2\sin(2t)$ and $y = t + 2\cos(5t)$ with t: $[-2\pi, 2\pi]$, x: $[-8, 8]$, y: $[-8, 8]$.

2. Given $x = f(t)$ and $y = g(t)$, what is $\frac{dy}{dx}$?

   From the Chain Rule, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

3. A good example: $x = 2\cos t - \sin t$ and $y = 2\sin t$ (Teardrop)

   a. Find equation of tangent line at $t = 0$.

   b. Find where graph has horizontal and vertical tangent lines.

4. Activity - The Ring of Fire.